

HYPERBOLIC HYDRA

N. BRADY, W. DISON AND T.R. RILEY

ABSTRACT. We give examples of hyperbolic groups with finite-rank free subgroups of huge (Ackermannian) distortion.

2010 Mathematics Subject Classification: 20F65, 20F10, 20F67

Key words and phrases: hyperbolic group, subgroup distortion, hydra, Ackermann's function

1. INTRODUCTION

1.1. Our result. Hyperbolic groups are algorithmically tractable (their word and conjugacy problem are straight-forward) and have the tree-like property that geodesic triangles in their Cayley graphs are close to tripods [5, 8]. The purpose of this article is to show that none-the-less some harbour extreme wildness within their subgroups — their finite-rank free subgroups, even. We prove (the terminology is explained below) —

Theorem 1.1. *For all $k \geq 1$ there is a hyperbolic group Γ_k with a free rank- $(k + 18)$ subgroup Λ_k whose distortion is $\text{Dist}_{\Lambda_k}^{\Gamma_k} \geq A_k$ — that is, it grows at least as fast as the k -th of the family of Ackermann functions.*

A *distortion function* measures the degree to which a subgroup folds in on itself within the ambient group. Suppose H is a subgroup with generating set T inside a group G with finite generating set S . Then $\text{Dist}_H^G : \mathbb{N} \rightarrow \mathbb{N}$ compares the intrinsic word metric d_T on H with the extrinsic word metric d_S :

$$\text{Dist}_H^G(n) := \max \{ d_T(1, g) \mid g \in H \text{ with } d_S(1, g) \leq n \}.$$

Up to the following equivalence, capturing qualitative agreement of growth rates, Dist_H^G does not depend on S and T . For $f, g : \mathbb{N} \rightarrow \mathbb{N}$, we write $f \leq g$ when there exists $C > 0$ such that $f(n) \leq Cg(Cn + C) + Cn + C$ for all n . Defining $f \simeq g$ when $f \leq g$ and $g \leq f$.

Ackermann's $A_k : \mathbb{N} \rightarrow \mathbb{N}$ are a family of fast-growing functions defined recursively:

$$\begin{aligned} A_0(n) &= n + 2 \text{ for } n \geq 0, \\ A_k(0) &= \begin{cases} 0 & \text{for } k = 1 \\ 1 & \text{for } k \geq 2, \end{cases} \\ \text{and } A_{k+1}(n + 1) &= A_k(A_{k+1}(n)) \text{ for } k, n \geq 0. \end{aligned}$$

In particular, $A_1(n) = 2n$, $A_2(n) = 2^n$ and $A_3(n)$ is the n -fold iterated power of 2. They are representatives of the successive levels of the Grzegorzcz hierarchy of primitive recursive functions — see, for example, [14].

Date: January 20, 2013.

The first author acknowledges partially supported from NSF grant DMS-0906962.

1.2. Background. Other heavily distorted free subgroups of hyperbolic groups have been exhibited by Mitra [12]: for all k , he gives an example with a free subgroup of distortion like a k -fold iterated exponential function and, more extreme, an example where the number of iterations grows like $\log n$. Barnard, the first author, and Dani developed Mitra's constructions into more explicit examples that are also $\text{CAT}(-1)$ [3]. We are not aware of any example of a hyperbolic group with a finite-rank free subgroup of distortion exceeding that of our examples. Indeed, we are not even aware of a hyperbolic group with a *finitely presented* subgroup of greater distortion. The Rips construction, applied to a finitely presentable group with unsolvable word problem yields a hyperbolic (in fact, $C'(1/6)$ small-cancellation) group G with a finitely generated subgroup N such that Dist_N^G is not bounded above by any recursive function, but these N are not finitely presentable. (See [1, §3.4], [7, Corollary 8.2], [9, §3, 3.K'''] and [13].)

1.3. An outline of our approach. Our groups Γ_k are elaborations of the *hydra groups*

$$G_k = \langle a_1, \dots, a_k, t \mid t^{-1}a_1t = a_1, t^{-1}a_it = a_ia_{i-1} \ (\forall i > 1) \rangle$$

explored by the second and third authors in [6] — groups that are $\text{CAT}(0)$, free-by-cyclic, biautomatic, and can be presented with only one relator, and yet the subgroups $H_k := \langle a_1t, \dots, a_kt \rangle$ are free of rank k and their distortion grows like the k -th of Ackermann's functions: $\text{Dist}_{H_k}^{G_k} \simeq A_k$.

This extreme distortion stems from a phenomenon which can be described in terms of a re-imagining of Hercules' battle with the Lernaean Hydra. A *hydra* is a positive word w on the alphabet a_1, a_2, \dots . Hercules removes the first letter and then the creature regenerates in that each remaining a_i with $i > 1$ becomes a_ia_{i-1} and each remaining a_1 is unaffected. This repeats and Hercules triumphs when the hydra is reduced to the empty word. The number of steps is denoted $\mathcal{H}(w)$. (Each step encompasses the removal of the first letter and then regeneration.) For example, $\mathcal{H}(a_2^3) = 7$ —

$$a_2^3 \rightarrow (a_2a_1)^2 \rightarrow a_1a_2a_1^2 \rightarrow a_2a_1^3 \rightarrow a_1^3 \rightarrow a_1^2 \rightarrow a_1 \rightarrow \varepsilon.$$

In [6] it is shown that Hercules will be victorious whatever hydra he faces, but the number of strikes it takes can be huge: the functions \mathcal{H}_k defined by $\mathcal{H}_k(n) = \mathcal{H}(a_k^n)$ grow like Ackermann's functions: $\mathcal{H}_k \simeq A_k$.

The group G_k is not hyperbolic because it has the subgroup $\langle a_1, t \rangle \cong \mathbb{Z}^2$. We obtain Γ_k by combining G_k with another free-by-cyclic group, which is hyperbolic, in such a way that the interaction between the two occurs in a pair of relations that replace $t^{-1}a_1t = a_1$. This will allow the hydra effect to persist in Γ_k .

Like G_k , the group Γ_k is free-by-cyclic and will be presented as such. We will prove it is hyperbolic by translating to an alternative presentation for which the associated Cayley 2-complex is $\text{CAT}(0)$, and then we will argue that there are no isometrically embedded copies of \mathbb{R}^2 and appeal to the Flat Plane Theorem.

Whilst we will not call on it in this paper (as we will give the translation between the presentations explicitly), a result that lies behind how we came to our examples is that if a 2-complex admits an S^1 -valued Morse function all of whose ascending and descending links are trees, then its fundamental group is free-by-cyclic [2]. [The ascending link for our examples is visible in Figure 2 as the subgraph made up of all edges connecting pairs of negative vertices. The descending is that made up of all edges connecting pairs of positive vertices. Both are trees.]

1.4. The (lack of) upper bound on the distortion. It seems likely that $\text{Dist}_{\Lambda_k}^{\Gamma_k} \simeq A_k$, but we do not offer a proof that $\text{Dist}_{\Lambda_k}^{\Gamma_k} \leq A_k$. The proof in [6] that $\text{Dist}_{H_k}^{G_k} \leq A_k$ in [6] may guide a proof that $\text{Dist}_{\Lambda_k}^{\Gamma_k} \leq A_k$, but that proof is technical and how to carry it over to $\text{Dist}_{\Lambda_k}^{\Gamma_k}$ is not readily apparent. We are content to present here just the lower bound (which we believe is the more significant).

1.5. The organisation of this article. In Section 2 we give two presentations for Γ_k and prove they are equivalent. In Section 3 we use the link condition to show that the Cayley 2-complex associated to one of these presentations is $\text{CAT}(0)$ and then the Flat Plane Theorem to establish hyperbolicity of Γ_k . The other presentation places Γ_k in a class of free-by-cyclic groups which are the subject of Section 4: we show that for $k \geq 2$, they contain free subgroups of rank $k + 18$ and distortion $\simeq A_k$. (In the case $k = 1$, Theorem 1.1 is elementary: take Γ_1 to be a free group and Λ_1 to be Γ_1 .)

2. OUR EXAMPLES

2.1. A $\text{CAT}(0)$ presentation for Γ_k . This presentation P_k is well suited to establishing hyperbolicity (see Section 3) —

generators: $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_8, \gamma_1, \dots, \gamma_8, \sigma, \tau,$

relations:

$$\begin{aligned} \alpha_i^{-1} \tau \alpha_i &= \alpha_{i+1} \quad (1 \leq i \leq k), \\ \beta_i^{-1} \tau \beta_i &= \beta_{i+1} \quad (1 \leq i \leq 7), \quad \beta_8 \tau \beta_8^{-1} = \beta_1, \\ \gamma_i^{-1} \sigma \gamma_i &= \gamma_{i+1} \quad (1 \leq i \leq 7), \quad \gamma_8 \sigma \gamma_8^{-1} = \gamma_1, \\ \gamma_3 \beta_5 &= \beta_3 \gamma_5, \quad \alpha_1 \gamma_1 \tau = \tau \gamma_7 \alpha_1. \end{aligned}$$

One can encode P_k in diagrams such as in Figure 1 which gives the case $k = 6$. Each edge in the three labelled oriented trees (*LOTs* — see [10]) encodes a commutator relation — an edge labelled y from a vertex labelled x to a vertex labelled z corresponds to a relation $y^{-1}xy = z$. The square and hexagonal 2-cells represent the remaining two relations, $\gamma_3\beta_5 = \beta_3\gamma_5$ and $\alpha_1\gamma_1\tau = \tau\gamma_7\alpha_1$.

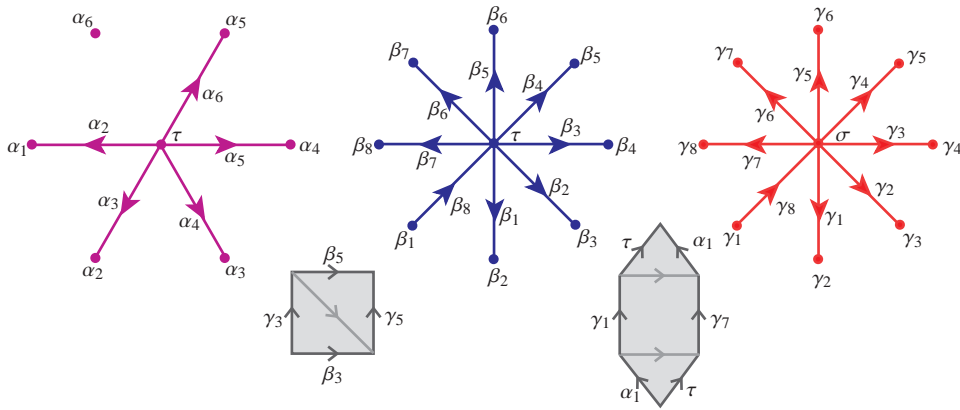


FIGURE 1. The defining relations of the presentation P_6 for Γ_6 displayed as three LOTs and two 2-cells.

If one removes the α_i and all the relations in which they appear from P_k , then one essentially gets groups studied by Meham & Muckerjee in [11]. These, in turn, are built from two copies of groups studied by Barnard and the first author in [2].

2.2. A presentation of Γ_k as a free-by-cyclic group. This presentation Q_k has —

generators: $a_0, \dots, a_k, b_1, \dots, b_8, c_1, \dots, c_8, d, t,$

relations:

$$\begin{aligned} t^{-1}a_it &= \theta(a_i) & 0 \leq i \leq k, & & t^{-1}c_it &= w^{-1}\psi(c_i)w & 1 \leq i \leq 8, \\ t^{-1}b_it &= \phi(b_i) & 1 \leq i \leq 8, & & t^{-1}dt &= \psi(c_3)\psi(c_5)\phi(b_5)^{-1}d\phi(b_3), \end{aligned}$$

where θ, ϕ and ψ are defined by

$$\begin{aligned} \theta(a_i) &= \begin{cases} ua_1v & i = 0, \\ a_0 & i = 1, \\ a_ia_{i-1} & 1 < i \leq k, \end{cases} \\ \phi(b_i) &= \begin{cases} b_i \dots b_8 & 1 \leq i \leq 7, \\ b_8^{-1} \dots b_1^{-1}b_8 & i = 8, \end{cases} \\ \psi(c_i) &= \begin{cases} c_i \dots c_8 & 1 \leq i \leq 7, \\ c_8^{-1} \dots c_1^{-1}c_8 & i = 8, \end{cases} \end{aligned}$$

and $u := t^{-k}c_7^{-1}c_3db_3^{-1}t^k$, $v := t^{-k}b_3d^{-1}c_3^{-1}c_1t^k$, and $w := \psi(c_3)^{-1}t^{-1}b_3d^{-1}t$.

Three remarks:

- (i) u, v , and w represent elements of the subgroup $\langle b_1, \dots, b_8, c_1, \dots, c_8, d \rangle$.
- (ii) Both ϕ and ψ define automorphisms of $F(b_1, \dots, b_8)$ and $F(c_1, \dots, c_8)$, respectively, as would θ for $F(a_0, \dots, a_k)$ were $\theta(a_0)$ equal to a_1 rather than ua_1v .
- (iii) The action of t by conjugation on

$$F(a_0, a_1, \dots, a_k, b_1, \dots, b_8, c_1, \dots, c_8, d)$$

apparent in the presentation Q_k is an automorphism because there are sequences of Nielsen transformations taking

$$\begin{aligned} &(a_0, a_1, \dots, a_k, b_1, \dots, b_8, c_1, \dots, c_8, d) \\ &\rightarrow (a_1, t^{-1}a_1t, \dots, t^{-1}a_kt, b_1, \dots, b_8, c_1, \dots, c_8, d) \\ &\rightarrow (t^{-1}a_0t, t^{-1}a_1t, \dots, t^{-1}a_kt, b_1, \dots, b_8, c_1, \dots, c_8, d) \\ &\rightarrow (t^{-1}a_0t, t^{-1}a_1t, \dots, t^{-1}a_kt, b_1, \dots, b_8, \psi(c_1), \dots, \psi(c_8), d) \\ &\rightarrow (t^{-1}a_0t, t^{-1}a_1t, \dots, t^{-1}a_kt, b_1, \dots, b_8, \psi(c_1), \dots, \psi(c_8), t^{-1}dt) \\ &\rightarrow (t^{-1}a_0t, t^{-1}a_1t, \dots, t^{-1}a_kt, b_1, \dots, b_8, t^{-1}c_1t, \dots, t^{-1}c_8t, t^{-1}dt) \\ &\rightarrow (t^{-1}a_0t, t^{-1}a_1t, \dots, t^{-1}a_kt, t^{-1}b_1t, \dots, t^{-1}b_8t, t^{-1}c_1t, \dots, t^{-1}c_8t, t^{-1}dt). \end{aligned}$$

The subgroup Λ_k of Theorem 1.1 will be

$$\langle a_0t, \dots, a_kt, b_1, \dots, b_8, c_1, \dots, c_8, d \rangle.$$

2.3. The equivalence of the presentations.

Proposition 2.1. P_k and Q_k present the same groups.

Before we prove this, let P'_k be the presentation obtained from P_k by removing all the generators a_i and all the relations in which they occur. Likewise, let Q'_k be that obtained from Q_k by removing all the generators a_i and all the relations in which they occur.

Lemma 2.2. *The groups presented by P'_k and Q'_k are isomorphic via*

$$\begin{aligned} \tau &\mapsto t^{-1}, & \beta_i &\mapsto \begin{cases} t^{-1}b_i & 1 \leq i \leq 7, \\ b_8 t^{-1} & i = 8, \end{cases} \\ \sigma &\mapsto s^{-1}, & \gamma_i &\mapsto \begin{cases} s^{-1}c_i & 1 \leq i \leq 7, \\ c_8 s^{-1} & i = 8, \end{cases} \end{aligned}$$

where $s = c_3 d b_3^{-1} t$.

Proof. As the reader can verify, $\tau \mapsto t^{-1}$, $\beta_i \mapsto t^{-1}b_i$ for $1 \leq i \leq 7$, and $\beta_8 \mapsto b_8 t^{-1}$ defines an isomorphism

$$\begin{aligned} \langle \beta_1, \dots, \beta_8, \tau \mid \beta_i^{-1} \tau \beta_i &= \beta_{i+1} \ (1 \leq i \leq 7), \quad \beta_8 \tau \beta_8^{-1} = \beta_1 \rangle \\ \rightarrow F(b_1, \dots, b_8) \rtimes_{\phi} \mathbb{Z} &= \langle b_1, \dots, b_8, t \mid t^{-1} b_i t = \phi(b_i) \ (1 \leq i \leq 8) \rangle. \end{aligned}$$

Likewise,

$$\begin{aligned} \langle \gamma_1, \dots, \gamma_8, \sigma \mid \gamma_i^{-1} \sigma \gamma_i &= \gamma_{i+1} \text{ for } 1 \leq i \leq 7, \quad \gamma_8 \sigma \gamma_8^{-1} = \gamma_1 \rangle \\ \cong \langle c_1, \dots, c_8, s \mid s^{-1} c_i s &= \psi(c_i) \text{ for } 1 \leq i \leq 8 \rangle, \end{aligned}$$

via $\sigma \mapsto s^{-1}$, $\gamma_i \mapsto s^{-1}c_i$ for $1 \leq i \leq 7$, and $\gamma_8 \mapsto c_8 s^{-1}$.

These maps translate the relation $\gamma_3 \beta_5 = \beta_3 \gamma_5$ to $s^{-1} c_3 t^{-1} b_5 = t^{-1} b_3 s^{-1} c_5$, which becomes $b_5^{-1} t d = c_5^{-1} c_3^{-1} d b_3^{-1} t$ on substituting for $s = c_3 d b_3^{-1} t$ and simplifying. Inverting, we find this is equivalent to $t b_5^{-1} t d = c_5^{-1} c_3^{-1} d b_3^{-1} t$, which rearranges to $t^{-1} d t = (t^{-1} c_3 c_5 b_5^{-1} t) d (t^{-1} b_3 t)$.

Now $c_3^{-1} s t^{-1} b_3 d^{-1} = 1$ since $s = c_3 d b_3^{-1} t$. So the relation $s^{-1} c_i s = \psi(c_i)$ is equivalent to $t^{-1} c_i t = w^{-1} \psi(c_i) w$ since

$$\begin{aligned} t^{-1} c_i t &= t^{-1} (d b_3^{-1} t s^{-1} c_3) c_i (c_3^{-1} s t^{-1} b_3 d^{-1}) t \\ &= (t^{-1} d t) (t^{-1} b_3^{-1} t) (s^{-1} c_3 s) (s^{-1} c_i s) (s^{-1} c_3^{-1} s) (t^{-1} b_3 t) (t^{-1} d^{-1} t) \\ &= w^{-1} \psi(c_i) w. \end{aligned}$$

□

Inductively define words u_i and v_i for $i \geq 0$ by

$$\begin{aligned} u_0 &= \alpha_k, & u_{i+1} &= u_i^{-1} t^{-1} u_i \quad (i \geq 0), \\ v_0 &= a_k, & v_{i+1} &= v_i^{-1} t^{-1} v_i \quad (i \geq 0). \end{aligned}$$

The following observation from [6] can be proved by inducting on i .

Lemma 2.3. *On substituting an a_k for each α_k in u_i , the words u_i and $t^{i-1} v_i t^{-i}$ become freely equal for all $i \geq 1$.*

Proof of Proposition 2.1. By Lemma 2.2 there is a sequence of Tietze moves carrying the subpresentation P'_k of P_k to Q'_k and the remaining relations (those involving the α_i) to

$$\alpha_1 s^{-1} c_1 t^{-1} = t^{-1} s^{-1} c_7 \alpha_1, \quad \alpha_i^{-1} t^{-1} \alpha_i = \alpha_{i-1}, \quad 1 < i \leq k.$$

A sequence of Tietze moves eliminating $\alpha_1, \dots, \alpha_{k-1}$ transforms this family to the single relation

$$u_{k-1} s^{-1} c_1 t^{-1} = t^{-1} s^{-1} c_7 u_{k-1}.$$

Now substitute an a_k for each α_k , then by Lemma 2.3, this relation is equivalent to

$$(t^{k-2} v_{k-1} t^{-(k-1)}) s^{-1} c_1 t^{-1} = t^{-1} s^{-1} c_7 (t^{k-2} v_{k-1} t^{-(k-1)}),$$

which becomes

$$t^{-1}v_{k-1}t = (t^{-(k-1)}c_7^{-1}st^{k-1})v_{k-1}(t^{-(k-1)}s^{-1}c_1t^{k-1})$$

on conjugating by t^{k-1} and rearranging. A sequence of Tietze moves introducing a_{k-1}, \dots, a_1 expands this to the family of relations

$$t^{-1}a_1t = t^{-(k-1)}c_7^{-1}st^{k-1}a_1t^{-(k-1)}s^{-1}c_1t^{k-1}, \quad t^{-1}a_it = a_ia_{i-1}, \quad 1 < i \leq k.$$

The first of these relations becomes $t^{-1}a_1t = a_0$ when we introduce a_0 together with the new relation

$$a_0 = t^{-(k-1)}c_7^{-1}st^{k-1}a_1t^{-(k-1)}s^{-1}c_1t^{k-1},$$

which becomes $t^{-1}a_0t = ua_1v$ on conjugating by t and eliminating the s and s^{-1} using $s = c_3db_3^{-1}t$. \square

3. HYPERBOLICITY

We establish hyperbolicity using techniques employed in [2] and [11].

Consider the presentation 2-complex K_k for P_k assembled from Euclidean unit-squares associated to each of the defining relations with the single exception of $\alpha_1\gamma_1\tau = \tau\gamma_1\alpha_1$ for which we use a Euclidean hexagon made from one unit square and two equilateral triangles as shown in Figure 1.

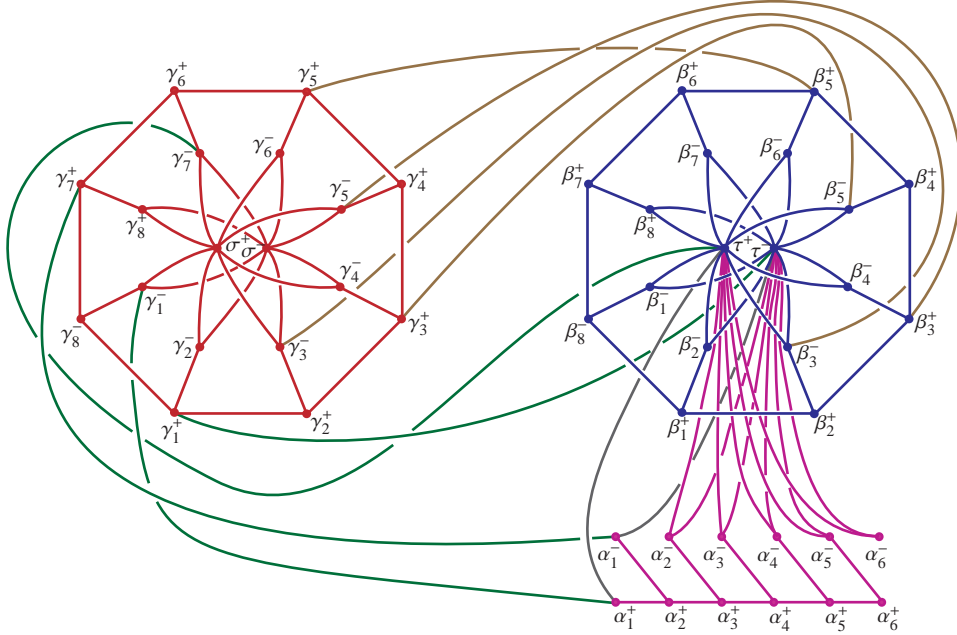


FIGURE 2. The link of the vertex in the presentation 2-complex associated to the presentation P_6 for Γ_6 given in Section 2. The two grey edges have length $\pi/3$, the four green edges have length $5\pi/6$, and all other edges have length $\pi/2$.

The link in the case $k = 6$ is shown in Figure 2. All edges have length $\pi/2$ apart from the edges $\tau^+ - \alpha_1^+$ and $\tau^- - \alpha_1^-$ (shown in grey), which have length $\pi/3$, and the edges from $\gamma_1^+ - \tau^+$, $\gamma_7^+ - \alpha_1^+$, $\alpha_1^+ - \gamma_1^+$, and $\tau^+ - \gamma_7^+$ (shown in green), which have length $5\pi/6$. Inspecting the link we see that any simple loop in the graph has length at least 2π (separately considering the cases of monochrome and multicoloured simple loops in Figure 1 helps to

check this) — that is, for all $k \geq 1$, the link is *large*. So K_k satisfies the *link condition* (see [5]) and its universal cover \widetilde{K}_k is therefore a CAT(0)–space.

To show that Γ_k is hyperbolic we will show that \widetilde{K}_k contains no subspace isometric to \mathbb{E}^2 and then appeal to the Flat Plane Theorem of [4, 8]. The link of a vertex in any isometric copy of \mathbb{E}^2 in \widetilde{K}_k would appear as a simple loop of length 2π in the link. But inspecting the link, we find that no edges of length $\pi/3$ or $5\pi/6$ (the grey and green edges) occur in a simple loop of length 2π . Next one can check the edges $\gamma_3^+ - \beta_5^-$, $\gamma_3^- - \beta_3^-$, $\gamma_5^- - \beta_3^+$ and $\gamma_5^+ - \beta_5^+$ (the brown edges in the figure) do not occur in a simple loop of length 2π . Then it becomes evident that edges occurring in simple loops of length 2π are precisely the edges

$$\begin{array}{llll} \tau^+ - \alpha_i^-, & \tau^- - \alpha_i^- & (2 \leq i \leq 7), \\ \tau^+ - \beta_i^-, & \tau^- - \beta_i^- & (1 \leq i \leq 7), & \tau^+ - \beta_8^+, \quad \tau^- - \beta_8^+, \\ \sigma^+ - \gamma_i^-, & \sigma^- - \gamma_i^- & (1 \leq i \leq 7), & \sigma^+ - \gamma_8^+, \quad \sigma^- - \gamma_8^+. \end{array}$$

So every corner of every 2–cell in an isometrically embedded copy of \mathbb{E}^2 must give rise to one of the edges in this list. But, looking at the defining relations, we see that no 2–cell in \widetilde{K}_k has this property. Therefore there are no such \mathbb{E}^2 and Γ_k is hyperbolic by the Flat Plane Theorem.

4. FREENESS AND DISTORTION

4.1. A family of free–by–cyclic groups. Fix an integer $l \geq 1$, words u and v on b_1, \dots, b_l , and an automorphism ϕ of $F(b_1, \dots, b_l)$. Then, for $k \geq 1$, define

$$\Psi_k := F(a_0, \dots, a_k, b_1, \dots, b_l) \rtimes_{\theta} \mathbb{Z}$$

where θ is the automorphism of $F(a_0, \dots, a_k, b_1, \dots, b_l)$ whose restriction to $F(b_1, \dots, b_l)$ is ϕ and

$$\theta(a_i) = \begin{cases} ua_1v & i = 0, \\ a_0 & i = 1, \\ a_i a_{i-1} & 1 < i \leq k. \end{cases}$$

Let t denote a generator of the \mathbb{Z} –factor, so $t^{-1}a_it = \theta(a_i)$ and $t^{-1}b_jt = \theta(b_j)$ for all i and j .

The presentation Q_k in Section 2.2 shows Γ_k is an example of such a Ψ_k .

Our aim in the remainder of this section is to establish —

Proposition 4.1. *The subgroup*

$$\Lambda_k := \langle a_0t, \dots, a_kt, b_1, \dots, b_l \rangle$$

of Ψ_k is free of rank $k + l + 1$ and $\text{Dist}_{\Lambda_k}^{\Psi_k} \simeq A_k$.

4.2. Towards a lower bound on distortion. In what follows, when, for a word $u = u(a_0, \dots, a_k, b_1, \dots, b_l)$, we refer to $\theta^r(u)$, we mean the freely reduced word that equals $\theta^r(u)$ in $F(a_0, \dots, a_k, b_1, \dots, b_l)$.

The extreme distortion in the hydra groups of [6] stemmed from the battle between Hercules and the hydra that we described in Section 1. When studying Ψ_k we will need the following more elaborate version of that battle. A *hydra* is now a word on $a_0, a_1, \dots, a_k, b_1, \dots, b_l$ in which the a_i only appear with positive exponents. As before, Hercules fights a hydra by removing the first letter. But in this version, the hydra only regenerates after an a_i is removed, and that regeneration is: each remaining a_i and $b_j^{\pm 1}$ becomes $\theta(a_i)$ and $\theta(b_j^{\pm 1})$, respectively. Again, we consider Hercules victorious if, on sufficient repetition, the hydra is reduced to the empty word.

Reprising the example from Section 1, Hercules defeats a_2^3 as follows:

$$\begin{aligned} a_2^3 &\rightarrow (a_2 a_1)^2 \rightarrow a_0 a_2 a_1 a_0 \rightarrow a_2 a_1 a_0 u a_1 v \rightarrow a_0 u a_1 v \theta(u) a_0 \theta(v) \\ &\rightarrow a_0 \theta(v) \theta^2(u) u a_1 v \theta^2(v) \rightarrow a_0 \theta(v) \theta^3(v) \rightarrow \varepsilon. \end{aligned}$$

— here, the steps in which Hercules removes a b_j are not shown; the arrows indicate the progression from when an a_i is about to be removed to when an a_i next appears at the front of the word or the hydra becomes the empty word.

The key point is that a_0 and the b_j play no essential role in this battle; if we removed all b_j and replaced all a_0 by a_1 , we would have a battle of the original form. Thus we have the following lemma. [Recall that $\mathcal{H}(u)$ denotes the duration of the battle (of the original type from Section 1 and [6]) against the hydra u .]

Lemma 4.2. *Hercules wins against all hydra w and, in the battle, the number of times he removes an a_i equals $\mathcal{H}(\bar{w})$ where \bar{w} is the word obtained from w by removing all $b_j^{\pm 1}$ and replacing all a_0 by a_1 .*

Consideration of the original battle between Hercules and the hydra led to the result that, for all $k, n \geq 1$, there is a positive word $u_{k,n} = u_{k,n}(a_1 t, \dots, a_k t)$ of length $\mathcal{H}_k(n)$ that equals $a_k^n t^{\mathcal{H}_k(n)}$ in G_k . (This is Lemma 5.1 in [6].) The reason is that the pairing off of a t with an initial a_i in a positive word on a_1, \dots, a_k corresponds to a decapitation, and the conjugation by t that moves that t into place from the right-hand end causes *regeneration* for the remainder of the word. For example $\mathcal{H}_2(3) = 7$ and

$$\begin{aligned} a_2^3 t^7 &= (a_2 t) t^{-1} a_2^2 t^6 = (a_2 t) (a_2 a_1)^2 t^6 = (a_2 t) (a_2 t) t^{-1} a_1 a_2 a_1 t^5 \\ &= \dots = (a_2 t) (a_2 t) (a_1 t) (a_2 t) (a_1 t) (a_1 t) (a_1 t) = u_{2,3}. \end{aligned}$$

In the corresponding calculation for Ψ_k , only the a_i get paired with t , and on each of the $\mathcal{H}_k(n)$ times that happens, the subsequent conjugation by t can increase length by a factor C which depends only on ϕ , $\ell(u)$ and $\ell(v)$. So —

Lemma 4.3. *There exists $C > 0$ such that for all $k, n \geq 1$, there is a word $\hat{u}_{k,n} = \hat{u}_{k,n}(a_0 t, \dots, a_k t, b_1, \dots, b_l)$ that equals $a_k^n t^{\mathcal{H}_k(n)}$ in Ψ_k and has the properties that*

$$\mathcal{H}_k(n) \leq \ell(\hat{u}_{k,n}) \leq C^{\mathcal{H}_k(n)} n$$

and all the $(a_i t)$ it contains have positive exponents.

This and our next two lemmas will be components of a calculation that will yield Proposition 4.6 (the analogue of Proposition 5.2 in [6]), which will be the key to establishing a lower bound on the distortion of Λ_k in Ψ_k .

A simple calculation yields —

Lemma 4.4. *$t^{-m} a_1 t^{m+1} = \tau_m$ in Ψ_k for all $m \geq 2$ where*

$$\tau_m := \begin{cases} a_0 t & \text{for } m = 1 \\ \phi^{m-2}(u) \dots \phi^2(u) u (a_1 t) \phi(v) \phi^3(v) \dots \phi^{m-1}(v) & \text{for even } m \geq 2 \\ \phi^{m-2}(u) \dots \phi^3(u) \phi(u) (a_0 t) \phi^2(v) \phi^4(v) \dots \phi^{m-1}(v) & \text{for odd } m > 2. \end{cases}$$

This combines with

$$t^{-1} (a_1 t^{-1})^n = (t^{-1} a_1 t^2) (t^{-3} a_1 t^4) (t^{-5} a_1 t^6) \dots (t^{1-2n} a_1 t^{2n}) t^{-2n-1}$$

to give —

Lemma 4.5. *There is a constant $C > 0$, depending only on ϕ , $\ell(u)$ and $\ell(v)$, such that for all $n \geq 0$, there is a word $v_n = v_n(a_0 t, a_1 t, b_1, \dots, b_l)$ such that $t^{-1} (a_1 t^{-1})^n = v_n t^{-2n-1}$ in Ψ_k , the number of $(a_1 t)$ contained in v_n is n and all have positive exponent, and $n \leq \ell(v_n) \leq C^n$.*

Proposition 4.6. *For all $k \geq 2$ and $n \geq 1$, there is a reduced word of length at least $2\mathcal{H}_k(n) + 3$ on $a_0t, a_1t, \dots, a_kt, b_1, \dots, b_l$, that equals $a_k^n a_2 t a_1 a_2^{-1} a_k^{-n}$ in Ψ_k .*

Proof. After rewriting the relation $t^{-1}a_2t = a_2a_1$ as $a_2^{-1}ta_2 = ta_1^{-1}$, we see that $a_2^{-1}t^{\mathcal{H}_k(n)}a_2 = (ta_1^{-1})^{\mathcal{H}_k(n)}$. So

$$a_k^n a_2 = \hat{u}_{k,n} a_2 (ta_1^{-1})^{-\mathcal{H}_k(n)}$$

for $\hat{u}_{k,n}$ as in Lemma 4.3. This gives the first of the equalities

$$\begin{aligned} a_k^n a_2 t a_1 a_2^{-1} a_k^{-n} &= \hat{u}_{k,n} a_2 (ta_1^{-1})^{-\mathcal{H}_k(n)} ta_1 (ta_1^{-1})^{\mathcal{H}_k(n)} a_2^{-1} \hat{u}_{k,n}^{-1} \\ &= \hat{u}_{k,n} (a_2 t) t^{-1} (ta_1^{-1})^{-\mathcal{H}_k(n)} ta_1 (ta_1^{-1})^{\mathcal{H}_k(n)} t (a_2 t)^{-1} \hat{u}_{k,n}^{-1} \\ &= \hat{u}_{k,n} (a_2 t) v_{\mathcal{H}_k(n)} t^{-2n-1} ta_1 t^{2n+1} v_{\mathcal{H}_k(n)}^{-1} (a_2 t)^{-1} \hat{u}_{k,n}^{-1} \\ &= \hat{u}_{k,n} (a_2 t) v_{\mathcal{H}_k(n)} \tau_{2n} v_{\mathcal{H}_k(n)}^{-1} (a_2 t)^{-1} \hat{u}_{k,n}^{-1}. \end{aligned}$$

The second is a free equality and the third and fourth are applications of Lemmas 4.5 and 4.4, respectively.

This calculation arrives at a word on $a_0t, a_1t, \dots, a_kt, b_1, \dots, b_l$, that equals $a_k^n a_2 t a_1 a_2^{-1} a_k^{-n}$ in Ψ_k . This word may not be freely reduced, but if we delete all the $b_j^{\pm 1}$ it contains, replace all $a_0^{\pm 1}$ by $a_1^{\pm 1}$, and then freely reduce (i.e. cancel away all $(a_it)^{\pm 1}(a_it)^{\mp 1}$ subwords), we get $u_{k,n} (a_2 t) (a_1 t) (a_2 t)^{-1} u_{k,n}^{-1}$, which has length $2\mathcal{H}_k(n) + 3$. \square

4.3. Freeness and rank. The result of this section is:

Proposition 4.7. *The subgroup Λ_k is free of rank $k + l + 1$.*

It will be convenient to prove more. In the special case where w represents the identity, the following proposition tells us that there are no non-trivial relations between $a_0t, \dots, a_kt, b_1, \dots, b_l$ and so establishes Proposition 4.7.

Proposition 4.8. *If $w = w(a_0t, \dots, a_kt, b_1, \dots, b_l)$ represents an element of the subgroup $\langle t \rangle$ in Ψ_k , then w freely equals the empty word.*

We begin with an observation on how the groups Ψ_k nest.

Lemma 4.9. *For $1 \leq i \leq k$, the canonical homomorphism $\Psi_i \rightarrow \Psi_k$ is an inclusion.*

Proof. The free-by-cyclic normal forms — a reduced word on $a_0, \dots, a_k, b_1, \dots, b_l$ times a power of t — of an element of Ψ_i and its image in Ψ_k are the same. \square

We will prove Proposition 4.8 by induction, but first we give a corollary which will be useful in the induction step. We emphasise that when we say that $v(a_0t, \dots, a_kt, b_1, \dots, b_l)$ is *freely reduced* in the following, we mean that there are no $(a_it)^{\pm 1}(a_it)^{\mp 1}$ or $b_j^{\pm 1}b_j^{\mp 1}$ subwords.

Corollary 4.10. *Suppose $v(a_0t, \dots, a_kt, b_1, \dots, b_l)$ is a freely reduced word equalling $\hat{v}t^s$ in Ψ_k where $s \in \mathbb{Z}$ and $\hat{v} = \hat{v}(a_0, \dots, a_k, b_1, \dots, b_l)$ is a word in which all the a_i that occur have positive exponents. Then all the (a_it) in v have positive exponents.*

Proof. The hydra battle described prior to Lemma 4.2, played out against $\hat{v}(a_0, \dots, a_k, b_1, \dots, b_l)$ gives a word $v' = v'(a_0t, \dots, a_kt, b_1, \dots, b_l)$ and an integer s' such that $v' = \hat{v}t^{s'}$ in Ψ_k . Moreover, the exponents of all the (a_it) in v' are positive. Now, $v^{-1}v' \in \langle t \rangle$ since $\hat{v} = vt^{-s} = v't^{-s'}$, and so v and v' are freely equal by Proposition 4.8. Therefore the exponents of all the (a_it) in v are positive. \square

Proof of Proposition 4.8. We induct on k . For the base case of $k = 1$, notice that defining $\bar{a}_0 := a_0 t$ and $\bar{a}_1 := a_1 t$, we can transform the presentation

$$\langle a_0, a_1, b_1, \dots, b_l, t \mid t^{-1} a_0 t = u a_1 v, t^{-1} a_1 t = a_0, t^{-1} b_j t = \phi(b_j) \forall j \rangle$$

for Ψ_1 to

$$\langle \bar{a}_0, \bar{a}_1, b_1, \dots, b_l, t \mid t^{-1} \bar{a}_0 t = u \bar{a}_1 \phi(v), t^{-1} \bar{a}_1 t = \bar{a}_0, t^{-1} b_j t = \phi(b_j) \forall j \rangle,$$

which is an alternative means of expressing Ψ_1 as a free-by-cyclic group from which the result is evident.

For the induction step, we consider a freely reduced word $w = w(a_0 t, \dots, a_k t, b_1, \dots, b_l)$ representing an element of $\langle t \rangle$ in Ψ_k where $k \geq 2$. If no $(a_k t)^{\pm 1}$ are present in w we can deduce from the induction hypothesis and Lemma 4.9 that w freely reduces to the empty word. For the remainder of our proof we suppose there are $(a_k t)^{\pm 1}$ present, and we seek a contradiction.

Consider shuffling the $t^{\pm 1}$ to the start of w using the defining relations — replacing each a_i and b_j passed by a $t^{\pm 1}$ with $\theta^{\pm 1}(a_i)$ and $\theta^{\pm 1}(b_j)$, respectively. The result will be a power of t times a word on $a_0, \dots, a_k, b_1, \dots, b_l$ which freely reduces to the empty word. Such is θ , no a_k are created or destroyed in this process of shuffling the $t^{\pm 1}$. So there is some expression $w_0(a_k t)^{\pm 1} u(a_k t)^{\mp 1} w_1$ for w such that $u = u(a_0 t, \dots, a_{k-1} t, b_1, \dots, b_l)$ and the $a_k^{\pm 1}$ and $a_k^{\mp 1}$ in the $(a_k t)^{\pm 1}$ and $(a_k t)^{\mp 1}$ buttressing u cancel after the shuffling and free reduction.

We will address first the case $w = w_0(a_k t)^{-1} u(a_k t) w_1$. Break down the shuffling process by first shuffling the $t^{\pm 1}$ out of w_0 , u and w_1 , and then carrying the resulting powers to the front of the word:

$$\begin{aligned} w &= w_0(a_k t)^{-1} u(a_k t) w_1 \rightarrow t^{r_0} \hat{w}_0(a_k t)^{-1} t^r \hat{u}(a_k t) t^{r_1} \hat{w}_1 \\ &\rightarrow t^{r_0+r+r_1} \theta^{r+r_1}(\hat{w}_0) \theta^{r+r_1+1}(a_k^{-1}) \theta^{r_1+1}(\hat{u}) \theta^{r_1+1}(a_k) \hat{w}_1 \end{aligned}$$

where $r_0, r, r_1 \in \mathbb{Z}$ and

$$\begin{aligned} \hat{w}_0 &= \hat{w}_0(a_0, \dots, a_k, b_1, \dots, b_l), \\ \hat{u} &= \hat{u}(a_0, \dots, a_{k-1}, b_1, \dots, b_l), \\ \hat{w}_1 &= \hat{w}_1(a_0, \dots, a_k, b_1, \dots, b_l) \end{aligned}$$

are words such that $t^{r_0} \hat{w}_0 = w_0$, $t^r \hat{u} = u$ and $t^{r_1} \hat{w}_1 = w_1$ in Ψ_k . When we expand $\theta^{r+r_1+1}(a_k^{-1})$ and $\theta^{r_1+1}(a_k)$ as words on a_0, \dots, a_k , the former ends with an a_k^{-1} which must cancel with the a_k at the start of the latter. So $\theta^{r_1+1}(\hat{u})$, and therefore \hat{u} , freely equal the empty word. So u represents an element of $\langle t \rangle$ and, by induction hypothesis, freely reduces to the empty word, contrary to the initial assumption that $w(a_0 t, \dots, a_k t, b_1, \dots, b_l)$ is reduced.

In the case $w = w_0(a_k t) u(a_k t)^{-1} w_1$, the shuffling process is

$$\begin{aligned} w &= w_0(a_k t) u(a_k t)^{-1} w_1 \rightarrow t^{r_0} \hat{w}_0(a_k t) t^r \hat{u}(a_k t)^{-1} t^{r_1} \hat{w}_1 \\ &\rightarrow t^{r_0+r+r_1} \theta^{r+r_1}(\hat{w}_0) \theta^{r+r_1}(a_k) \theta^{r_1-1}(\hat{u}) \theta^{r_1}(a_k^{-1}) \hat{w}_1 \end{aligned}$$

where $t^{r_0} \hat{w}_0 = w_0$, $t^r \hat{u} = u$ and $t^{r_1} \hat{w}_1 = w_1$ in Ψ_k , as before. The first and last letters of $\theta^{r+r_1}(a_k) \theta^{r_1-1}(\hat{u}) \theta^{r_1}(a_k^{-1})$ are a_k and a_k^{-1} which cancel, so this subword must freely reduce to the empty word. So $\theta^r(a_k) \theta^{-1}(\hat{u}) a_k^{-1}$ also freely reduces to the empty word — that is, $\theta^{r+1}(a_k) \hat{u}$ freely equals $a_k a_{k-1}$.

If $r = 0$ then this says that \hat{u} freely equals the empty word and, as before, we have a contradiction. Suppose $r > 0$. Then $\hat{u}^{-1} = (a_k a_{k-1})^{-1} \theta^{r+1}(a_k)$ would be a positive word on a_0, \dots, a_{k-1} were we to remove all the $b_1^{\pm 1}, \dots, b_l^{\pm 1}$ it contains. So, as $\hat{u}^{-1} t^{-r} = u^{-1}$, Corollary 4.10 applies and tells us that u^{-1} would be a positive word were we to remove all the $b_1^{\pm 1}, \dots, b_l^{\pm 1}$ it contains. But r is the exponent sum of the $(a_0 t)^{\pm 1}, \dots, (a_{k-1} t)^{\pm 1}$ in

u , and so we deduce the contradiction $r \leq 0$. Finally we note that the case $r < 0$ also leads to a contradiction because if we replace w by w^{-1} it becomes the case $r > 0$. \square

4.4. Conclusion. We deduce from Proposition 4.8 that the word posited in Proposition 4.6 is the *unique* reduced word on $a_0t, \dots, a_kt, b_1, \dots, b_l$ that equals $a_k^n a_2 t a_1 a_2^{-1} a_k^{-n}$ in Ψ_k . This establishes that $\text{Dist}_{\Lambda_k}^{\Psi_k} \geq \mathcal{H}_k$ for all $k \geq 2$. So, by Proposition 1.2 in [6], which says that $\mathcal{H}_k \simeq A_k$ for all $k \geq 1$, we have $\text{Dist}_{\Lambda_k}^{\Psi_k} \geq A_k$ for all $k \geq 2$. Added to Proposition 4.7, this completes the proof of Proposition 4.1.

Proposition 4.1 applies to the subgroup

$$\langle a_0t, \dots, a_kt, b_1, \dots, b_8, c_1, \dots, c_8, d \rangle$$

of Γ_k (presented as Q_k of Section 2) and so, as we established Γ_k to be hyperbolic in Section 3, Theorem 1.1 is proved.

REFERENCES

- [1] G. Arzhantseva and D. Osin. Solvable groups with polynomial Dehn functions. *Trans. Amer. Math. Soc.*, 354(8):3329–3348, 2002.
- [2] J. Barnard and N. Brady. Distortion of surface groups in CAT(0) free-by-cyclic groups. *Geom. Dedicata*, 120:119–139, 2006.
- [3] J. Barnard, N. Brady, and P. Dani. Super-exponential distortion of subgroups of CAT(−1) groups. *Algebr. Geom. Topol.*, 7:301–308, 2007.
- [4] M. R. Bridson. On the existence of flat planes in spaces of nonpositive curvature. *Proc. Amer. Math. Soc.*, 123(1):223–235, 1995.
- [5] M. R. Bridson and A. Haefliger. *Metric Spaces of Non-positive Curvature*. Number 319 in Grundlehren der mathematischen Wissenschaften. Springer Verlag, 1999.
- [6] W. Dison and T. R. Riley. Hydra groups. [arXiv:1002.1945](https://arxiv.org/abs/1002.1945).
- [7] B. Farb. The extrinsic geometry of subgroups and the generalised word problem. *Proc. London Math. Soc.* (3), 68(3):577–593, 1994.
- [8] M. Gromov. Hyperbolic groups. In S. M. Gersten, editor, *Essays in group theory*, volume 8 of *MSRI publications*, pages 75–263. Springer–Verlag, 1987.
- [9] M. Gromov. Asymptotic invariants of infinite groups. In G. Niblo and M. Roller, editors, *Geometric group theory II*, number 182 in LMS lecture notes. Camb. Univ. Press, 1993.
- [10] J. Howie. On the asphericity of ribbon disc complements. *Trans. Amer. Math. Soc.*, 289(1):281–302, 1985.
- [11] T. Meham and A. Mukherjee. Hyperbolic groups which fiber in infinitely many ways. *Algebr. Geom. Topol.*, 9(4):2101–2120, 2009.
- [12] M. Mitra. Cannon–Thurston maps for trees of hyperbolic metric spaces. *J. Diff. Geom.*, 48(1):135–164, 1998.
- [13] Ch. Pittet. *Géométrie des groupes, inégalités isopérimétriques de dimension 2 et distorsions*. PhD thesis, Université de Genève, 1992.
- [14] H. E. Rose. *Subrecursion: functions and hierarchies*, volume 9 of *Oxford Logic Guides*. The Clarendon Press Oxford University Press, New York, 1984.

NOEL BRADY

Department of Mathematics, Physical Sciences Center, 601 Elm Ave, University of Oklahoma, Norman, OK 73019, USA
 nbrady@math.ou.edu, <http://aftermath.math.ou.edu/~nbrady/>

WILL DISON

Bank of England, Threadneedle Street, London, EC2R 8AH, UK
 william.dison@gmail.com, <http://www.maths.bris.ac.uk/~mawjd/>

TIMOTHY R. RILEY

Department of Mathematics, 310 Malott Hall, Cornell University, Ithaca, NY 14853, USA
 tim.riley@math.cornell.edu, <http://www.math.cornell.edu/~riley/>